On Jacobi's Theorem respecting the relative equilibrium of a Revolving Ellipsoid of Fluid, and on Ivory's discussion of the Theorem." By I. TODHUNTER, M.A., F.R.S., late Fellow of St. John's College, Cambridge. Received November 23, 1869*.

1. The late James Ivory contributed to the Philosophical Transactions various memoirs on the subject of the equilibrium of fluids and the figure of the earth: the memoirs will be found in the volumes for 1824, 1831, 1834, and 1839. Ivory objected to the received theory of the equilibrium of fluids, and advocated some peculiar opinions at great length, and with much repetition. I do not propose now to criticise these memoirs; I will merely state that I consider them to be altogether unsatisfactory.

2. There is, however, one theorem in the general subject to which I now propose to draw attention, namely, Jacobi's theorem respecting the possibility of the relative equilibrium of an ellipsoid of fluid having three unequal axes and revolving about the least. Ivory discussed this theorem, and his errors are so numerous and so singular, that I have thought it would be desirable to place the corrections before the Society which originally received and published Ivory's communications. In correcting Ivory's errors and supplying his defects, I shall add something to the discussions which have hitherto been given of the theorem itself. It will be seen as we proceed that one of Ivory's errors has been already noticed and corrected.

3. Ivory first alluded to the matter in the memoir of 1834, which was read to the Royal Society a few months before Jacobi announced his discovery of the theorem. Of course at that date Ivory held the common opinion, that the relative equilibrium of a revolving ellipsoid with three unequal axes was impossible. But he does not merely acquiesce in the erroneous opinion, he attempts to *demonstrate* it in the following manner:—

"Further, the figure of the fluid in equilibrium can be no other than a spheroid of revolution. Draw a plane through the axis of rotation and any point (xyz) in the surface of the fluid. This plane will contain that part of the attraction of the spheroid which is parallel to the axis of rotation, or to the coordinate x: it will also contain the centrifugal force directed at right angles from the axis of rotation. *The same plane will also contain the resultant of the attractions parallel to y and z*; for if it did not, the resultant might be resolved into two forces, one contained in the plane, and the other perpendicular to it; and the force perpendicular to the plane would partly act in a direction touching the surface of the spheroid, which is inconsistent with the equilibrium of the fluid. Wherefore, the whole attractive force at any point in the surface of the spheroid is contained in a plane passing through the point and the axis of rotation;

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* Read Jan. 20, 1870. See vol. xviii. p. 171.

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which obviously excludes ellipsoids with three unequal axes, and limits the figures of equilibrium to spheroids formed by the revolution of an ellipsis about the axis of rotation; . . ."

The error here begins with the sentence which I have put in italics; the resultant of the attractions parallel to y and z need not act in the plane which Ivory specifies: the component which he obtains in a plane touching the surface may be balanced by a like component arising from the attraction parallel to x and the the so-called centrifugal force.

4. To the Philosophical Transactions for 1838 Ivory contributed a memoir of ten pages on Jacobi's theorem. Ivory devotes a few sentences to the history of the matter. He records the fact that Lagrange had inferred that the figure of relative equilibrium must be a figure of revolution. He makes no allusion, however, to his own erroneous demonstration in the volume for 1834.

5. The object of the memoir seems to be twofold—to establish Jacobi's theorem, and to deduce numerical results relating to the extreme possible cases analogous to those which had long been known relating to the extreme possible cases for an ellipsoid of revolution. The first object is attained; Jacobi's theorem is demonstrated in a manner resembling that which had previously been used by Liouville. The second object Ivory fails to attain, owing to an error in his process.

6. In the second page of the memoir there is an error in mechanics resembling that which we have already noticed in Art. 3. At any point in the surface of an ellipsoid, let the normal to the surface be drawn; and let it be terminated by the principal plane which is perpendicular to the axis of rotation: let ρ be the length of this straight line. At the same point in the surface draw a straight line in the direction of the resultant of the attraction of the whole mass of the ellipsoid, and let it be terminated by the same principal plane; let ρ' be the length of this straight line, then Ivory says:—

"Let σ denote the third side of the triangle which has ρ and ρ' for its other sides: then σ will represent the only force which, together with the attractive force ρ', will produce a resultant in the direction of ρ at right angles to the surface of the ellipsoid."

This statement is quite wrong. Any straight line which is in the same plane as the normal at any point, and the direction of the resultant attraction at that point, may be taken for the *direction* of such an additional force as Ivory requires; and the *magnitude* of the force can then be properly determined.

7. In order to render the discussion of Ivory's memoir readily intelligible, it will be necessary to indicate briefly the demonstration of Jacobi's theorem.

Let the equation to an ellipsoid be

$$x^2+\frac{y^2}{1+\lambda^2}+\frac{z^2}{1+\mu^2}=k^2,$$

The attractions which the ellipsoid exerts on a point (x, y, z) parallel to the axes of coordinates are known to be respectively Ax, By, Cz, where

$$A=\frac{3M}{k^3}\int_0^1\frac{u^2du}{H},\ B=\frac{3M}{k^3}\int_0^1\frac{u^2du}{(1+\lambda^2u^2)H},\ C=\frac{3M}{k^3}\int_0^1\frac{u^2du}{(1+\mu^2u^2)H}.$$

Here M denotes the mass of the ellipsoid, and H stands for

$$\sqrt{(1+\lambda^2u^2)\,(1+\mu^2u^2)}:$$

see the 'Mécanique Céleste,' Livre III. No. 7.

Take the axis of x for that of revolution, and let ω be the angular velocity. Then the necessary and sufficient condition for relative equilibrium is that the equation

$$Axdx+(B-\omega^2)ydy+(C-\omega^2)zdz=0$$

should coincide with the differential equation to the surface of the ellipsoid, namely,

$$xdx+\frac{ydy}{1+\lambda^2}+\frac{zdz}{1+\mu^2}=0.$$

Hence we have the conditions

$$\frac{B-\omega^2}{A}=\frac{1}{1+\lambda^2},\quad \frac{C-\omega^2}{A}=\frac{1}{1+\mu^2}.\quad \ldots\ldots\ldots \quad (1)$$

If we eliminate ω^2 between these we obtain

$$(1+\lambda^2)\,(1+\mu^2)\,(B-C)=A(\mu^2-\lambda^2).\ \ldots\ldots \quad (2)$$

But from the values of B and C we have

$$B-C=\frac{3M}{k^3}(\mu^2-\lambda^2)\int_0^1\frac{u^4du}{H^3}.$$

Thus (2) becomes

$$(\mu^2-\lambda^2)\left\{(1+\lambda^2)\,(1+\mu^2)\int_0^1\frac{u^4du}{H^3}-\int_0^1\frac{u^2du}{H}\right\}=0.$$

This may be satisfied by putting $\mu^2=\lambda^2$, which gives us an ellipsoid of revolution. Or it may be satisfied by making

$$(1+\lambda^2)\,(1+\mu^2)\int_0^1\frac{u^4du}{H^3}=\int_0^1\frac{u^2du}{H};$$

this may be reduced to

$$\int_0^1\frac{u^2(1-u^2)\,(1-\lambda^2\mu^2u^2)du}{H^3}=0.\quad \ldots\ldots \quad (3)$$

We have then to show that for suitable values of λ and μ this relation will hold. This will be shown in the sequel. We must also show that the value of ω^2 is positive. From (1) we have

$$\frac{B-\omega^2}{C-\omega^2}=\frac{1+\mu^2}{1+\lambda^2},$$

so that

$$\omega^2 = \frac{B-C}{\lambda^2-\mu^2} + \frac{B\lambda^2 - C\mu^2}{\lambda^2-\mu^2}.$$

Put in the values of B and C, and this reduces to

$$\omega^2 = \frac{3M}{k^3}\int_0^1 \frac{u^2(1-u^2)du}{H^3},$$

so that ω^2 is positive.

8. Before considering Jacobi's theorem, we will advert to the case in which $\mu^2 = \lambda^2$.

We have from (1)

$$\omega^2 = B - \frac{A}{1+\lambda^2};$$

let ρ be the density of the ellipsoid, then

$$M = \frac{4\pi\rho k^3(1+\lambda^2)}{3}.$$

Put q for $\frac{\omega^2}{\frac{4\pi\rho}{3}}$; thus

$$q = 3l^2\int_0^1 \frac{x^2(1-x^2)dx}{(1+l^2x^2)^2}. \quad \ldots \ldots \quad (4)$$

Here we have changed u into x, and λ into l, in order to have the same notation as Ivory has. Ivory says:

"From the equation (4) we learn that q will be known when l is given, or that every spheroid of a determinate form requires an appropriate velocity of rotation.

"The inspection of the same equation is sufficient to show that q is positive for all values of l^2; and as it vanishes both when l^2 is zero and infinitely great, it must pass at least once from increasing to decreasing, or it will admit of at least one maximum value. By differentiating with regard to l we obtain

$$\frac{dq}{2ldl} = \int_0^1 \frac{3x^2(1-x^2)(1-l^2x^2)dx}{(1+l^2x^2)^3}, \quad \ldots \ldots \quad (5)$$

from which formula we learn that $\frac{dq}{2ldl}$ is positive between the limits $l^2=0$ and $l^2=1$; that it will consist of a positive and a negative part when l^2 is greater than 1; and the positive part decreasing *while the negative part increases,* that it will ultimately be negative when l^2 is infinitely great. It follows therefore that $\frac{dq}{2ldl}$ can be only once equal to zero, and consequently that q can have only one maximum value, while l^2 increases from 0 to ∞."

This is quite unsound, because the words which I have put in italics are untrue. There are two ways of separating the integral into a positive part and a negative part. We may take for the positive part the integral between the limits 0 and $\frac{1}{l}$, and for the negative part the integral between the limits $\frac{1}{l}$ and 1. Or we may put the integral in the form

$$3\int_0^1 \frac{x^2(1-x^2)dx}{(1+l^2x^2)^3} - 3l^2\int_0^1 \frac{x^4(1-x^2)dx}{(1+l^2x^2)^3}.$$

I do not know which of these two ways Ivory adopted. It is true that in each of them the positive part decreases as l increases; but in each of them the negative part is finite when l is finite, and is infinitesimal when l is infinite, and so does not always increase with l as Ivory supposes.

9. However, although Ivory's reasoning is unsound, the result which he wishes to establish is correct, as I shall now show.

Consider the integral $\int_0^1 \frac{x^2(1-x^2)(1-l^2x^2)dx}{(1+l^2x^2)^3}$. We have to show that as l increases from unity to infinity, the integral vanishes and changes sign once, and *only once.*

It is obvious that the integral must vanish once, because it is positive when $l=1$, and negative when $l=\infty$: it is indeed infinitesimal in the latter case, but the sign is certainly negative.

Put z for lx; thus the integral becomes

$$\frac{1}{l^5}\int_0^l \frac{z^2(l^2-z^2)(1-z^2)dz}{(1+z^2)^3} = \frac{u}{l^5}, \text{ say.}$$

We have to show that the equation $u=0$ can have only one root between $l=1$ and $l=\infty$. If the equation could have more than one root it must have three roots at least; and then the equation $\frac{du}{dl}=0$ must have two roots at least. Now

$$\frac{du}{dl} = 2l\int_0^l \frac{z^2(1-z^2)dz}{(1+z^2)^3} = 2lv, \text{ say,}$$

and

$$\frac{dv}{dl} = \frac{l^2(1-l^2)}{(1+l^2)^3}.$$

Thus v is positive when $l=1$, and while l increases $\frac{dv}{dl}$ is always negative; thus v continually diminishes algebraically, and so cannot change sign more than once. Hence $\frac{du}{dl}$ cannot vanish more than once; it will vanish once because v is positive when $l=1$, and has a finite negative value when $l=\infty$, namely,

$$\int_0^{\frac{\pi}{2}} \sin^2\theta(1-2\sin^2\theta)d\theta, \text{ that is } -\frac{\pi}{8}.$$

10. There is also another way in which the result may be established. For from (4), by the known method of integration, we have

$$q=\frac{3(3+l^2)}{2l^3}\tan^{-1}l-\frac{9}{2l^2};$$

therefore

$$-\frac{dq}{dl}=\frac{3(9+l^2}{2l^4}\tan^{-1}l-\frac{3(9+7l^2)}{2(1+l^2)l^3}.$$

Thus $\frac{dq}{dl}$ vanishes when

$$\tan^{-1}l=\frac{9l+7l^3}{(1+l^2)(9+l^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Ivory gives these formulæ with some misprints.

Now it has been shown by Laplace that the equation (6) has one positive root, and only one, namely, when $l=2{\cdot}529\ldots.$ See the 'Mécanique Céleste,' Livre III. No. 20.

11. We now return to Jacobi's theorem. We take the integral in (3), and put x for u; we put $\lambda-\mu=\tau$, and we may suppose λ greater than μ, so that τ is positive; and we put $\lambda\mu=p$. Thus (3) becomes

$$\int_0^1\frac{x^2(1-x^2)(1-p^2x^2)dx}{\{(1+px^2)^2+\tau^2x^2\}^{\frac{3}{2}}}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Denote the left-hand member of (7) by V; then we propose to examine the range of values of p and τ which make V vanish, supposing both p and τ positive. If we regard p as an abscissa, and τ as an ordinate corresponding to p, we in effect propose to trace in the first quadrant the curve determined by the equation $V=0$.

12. If we put $\tau=0$, the equation (7) becomes

$$\int_0^1\frac{x^2(1-x^2)(1-p^2x^2)dx}{(1+px^2)^3}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Ivory says: "It is obvious that there is only one value of p that will verify the equation just found; for the integral can pass only once from being positive to be negative while p increases from 1 to be infinitely great."

I cannot admit that this assertion is obvious; the result, however, may be established by the following investigation:—

Denote the integral by u; as long as p is less than unity, u is positive, and when p is infinite u is negative. Thus u must vanish once as p changes from unity to infinity: we have to show that u can vanish only once.

If the equation $u=0$ could have more than one root, it must have three roots at least; and then the equation $\frac{du}{dp}=0$ must have two roots at least: this we shall show to be impossible.

We have

$$\frac{du}{dp}=-\int_0^1\frac{x^4(1-x^2)(3+2p-p^2x^2)dx}{(1+px^2)^4};$$

it is obvious then that $\frac{du}{dp}$ cannot vanish until $\frac{p^2}{3+2p}$ is greater than unity, that is, until p is greater than 3. Thus $\frac{du}{dp}$ is negative when $p=3$; and we can see that $\frac{du}{dp}$ is positive when $p=\infty$. Hence $\frac{du}{dp}$ changes sign once as p increases from 3 to infinity.

Put $px^2 = \tan^2\theta$, and let $p = \tan^2\beta$; thus

$$\frac{du}{dp} = \frac{1}{p^{\frac{5}{2}}} \int_0^\beta \frac{\sin^4\theta}{\cos^2\theta}(p\cos^2\theta - \sin^2\theta) \left\{ 1 - \left(3 + \frac{3}{p}\right) \cos^2\theta \right\} d\theta = \frac{v}{p^{\frac{5}{2}}}, \text{ say.}$$

Then we know that v is negative when $p=3$, and positive when $p=\infty$. If, then, the equation $v=0$ could have more than one root, it must have three roots at least, and then the equation $\frac{dv}{dp}=0$ must have two roots at least. Now

$$\frac{dv}{dp} = \int_0^\beta \sin^4\theta\left(1 - 3\cos^2\theta - \frac{3\sin^2\theta}{p^2}\right) d\theta,$$

$$\frac{d^2v}{dp^2} = \frac{6}{p^3}\int_0^\beta \sin^6\theta d\theta + \frac{(p-3)\sqrt{p}}{2(1+p)^3}.$$

Thus $\frac{d^2v}{dp^2}$ is always positive when p is greater than 3, and therefore $\frac{dv}{dp}$ continually increases, and so cannot vanish more than once.

Hence v cannot vanish more than once, and therefore u cannot vanish more than once. Thus u is always negative when p has any value greater than that which makes u vanish.

13. There is also another way in which the result may be established. It will be found that

$$\int_0^1 \frac{x^2(1-x^2)(1-p^2x^2)dx}{(1+px^2)^3} = \frac{3+13p}{8p^2} - \frac{3+14p+3p^2}{8p^{\frac{5}{2}}} \tan^{-1}\sqrt{p}.$$

Then it may be shown that the last expression will vanish once, and only once, as p changes from 1 to ∞. This method is adopted by Liouville in an article in his 'Journal de Mathématiques' for April 1839: the article consists of observations on the memoir by Ivory, which we are discussing.

Liouville says that the value of p, which makes the last expression vanish, is a little less than 2. Ivory has a formula which is equivalent to this, but he does not employ it to show that there is only one value of p; for he had already, as we have seen, stated this to be obvious. From the circumstance that Liouville gives a strict demonstration, it is plain that he agrees with me in thinking that Ivory's statement is not obvious.

According to Ivory, the value of p which satisfies (8) is 1·9414. . . . We will denote this by p_0.

14. I shall now show that if we ascribe to p any value greater than p_0, a corresponding value of r exists, which will make V vanish.

Let such a value be ascribed to p; then from Art. 12 it follows that, with this value of p and with $\tau=0$, the value of V will be negative: we shall now show that by taking τ large enough V will be positive,

$$V=\int_0^1 \frac{x^2(1-x^2)(1-p^2x^2)dx}{\{1+(2p+\tau^2)x^2+p^2x^4\}^{\frac{3}{2}}};$$

thus the sign of V is the same as that of

$$\int_0^1 \frac{x^2(1-x^2)(1-p^2x^2)dx}{(c^2+x^2+p^2c^2x^4)^{\frac{3}{2}}},$$

where c^2 stands for $\frac{1}{2p+\tau^2}$.

When c is made small enough, the term $p^2c^2x^4$ may be neglected in comparison with x^2; and so the sign of the integral will become the same as that of

$$\int_0^1 \frac{x^2(1-x^2)dx}{(c^2+x^2)^{\frac{3}{2}}}-\int_0^1 \frac{p^2x^4(1-x^2)dx}{(c^2+x^2)^{\frac{3}{2}}}.$$

The first term is infinite when $c=0$, and the second term is finite; thus the sign of the integral is positive when c is small enough. Since V is negative when $\tau=0$, and is positive when τ is large enough, it must vanish for some intermediate value of τ.

15. Moreover, when c is very small, we have approximately

$$\int_0^1 \frac{x^2(1-x^2)dx}{(c^2+x^2)^{\frac{3}{2}}}=\log\frac{2}{c}-\frac{3}{2}=\log\frac{2}{ce\sqrt{e}}.$$

$$\int_0^1 \frac{p^2x^4(1-x^2)dx}{(c^2+x^2)^{\frac{3}{2}}}=\frac{p^2}{4}.$$

Thus to make V vanish when τ is *very large*, we have approximately

$$\frac{2}{ce\sqrt{e}}=e^{\frac{p^2}{4}},$$

therefore

$$2(2p+\tau^2)^{\frac{1}{2}}=e^{\frac{p^2}{4}+\frac{3}{2}};$$

so that approximately

$$\tau=\tfrac{1}{2}e^{\frac{p^2}{4}+\frac{3}{2}}.$$

16. We shall next show that corresponding to a given value of p there is *only one value* of τ which will make V vanish.

Put

$$\Delta^2=(1+px^2)^2+\tau^2x^2,$$

$$P^2=(1+p)^2+\tau^2;$$

then

$$(1-x^2)(1-p^2x^2)=\Delta^2-P^2x^2.$$

Thus we have

$$V=\int_0^1 \frac{x^2dx}{\Delta}-P^2\int_0^1 \frac{x^4dx}{\Delta^3};$$

and therefore

$$\frac{dV}{\tau d\tau} = -3\int_0^1 \frac{x^4dx}{\Delta^3} + 3P^2\int_0^1 \frac{x^6dx}{\Delta^5}.$$

Hence

$$V + \frac{P^2}{3}\,\frac{dV}{\tau d\tau} = \int_0^1 \frac{x^2dx}{\Delta}\left(1-\frac{P^2x^2}{\Delta^2}\right)^2 dx. \quad . \quad . \quad . \quad . \quad (9)$$

The right-hand member is necessarily positive. If we put $\sigma=\tau^2$, we may write the equation thus,

$$\frac{d}{d\sigma}(V.P^3) = \frac{3}{2}P\int_0^1 \frac{x^2dx}{\Delta}\left(1-\frac{P^2x^2}{\Delta^2}\right)^2 dx.$$

This shows that if p be kept constant, VP^3 continually increases with σ, and therefore cannot vanish more than once, and therefore of course V cannot vanish more than once.

17. Ivory makes the following statement:—

"Let V stand for the integral in the equation (7); and supposing that p and τ^2 vary so as always to satisfy that equation, we shall have

$$\frac{dV}{dp}dp + \frac{dV}{\tau d\tau}\tau d\tau = 0.$$

Now, τ^2 representing any positive quantity, we may conceive it to increase from zero to be infinitely great; in which case it follows from the nature of the function V, that during the whole increase $\frac{dV}{\tau d\tau}\tau d\tau$ will be negative: wherefore the other term $\frac{dV}{dp}dp$ will be positive; which requires that p decrease continually."

It is here in fact asserted that $\frac{dV}{d\tau}$ is negative for such values of p and τ as make V vanish; this is, however, wrong, as equation (9) shows that $\frac{dV}{d\tau}$ is positive when $V=0$. The mistake was pointed out by Liouville in the memoir already cited, and the correction was accepted by Ivory in the Philosophical Transactions for 1839, pages 265, 266. The mistake vitiates the remainder of Ivory's memoir.

The investigation of Art. 16 is taken in substance from Liouville's memoir; he also demonstrates the proposition of Art. 14, but not in the way which I have adopted.

18. The extract given in the preceding article from Ivory's memoir involves another error, which Liouville does not notice: the words "which requires that p decrease continually," contain an arbitrary unproved assertion. We have

$$\frac{dV}{dp}\,\frac{dp}{d\tau} + \frac{dV}{d\tau} = 0;$$

hence if $\frac{dV}{d\tau}$ were negative, $\frac{dV}{dp}\,\frac{dp}{d\tau}$ would be positive; then $\frac{dp}{d\tau}$ might be positive or it might be negative: we cannot assert at once, as Ivory does, that $\frac{dp}{d\tau}$ must be negative. Suppose, for example, that V stood for $p^2-\tau^2-1$; then $\frac{d\text{V}}{d\tau}$ would be negative for positive values of τ, and p would continually increase with τ.

19. As I have already stated, Ivory accepted the correction made by Liouville; but in the two pages in which the mistake is acknowledged other untenable assertions are advanced. It was in effect necessary for Ivory's purpose to trace the curve determined by $\text{V}=0$ in the first quadrant; but instead of *demonstration* such as we have supplied in Art. 14, Ivory gives unwarranted assertions of the kind already noticed in Art. 18.

We have

$$\frac{d\text{V}}{dp}=-\int_0^1\frac{x^4(1-x^2)}{\Delta^5}\{(3+2p-p^2x^2)(1+px^2)+2p\tau^2x^2\}dx,$$

from which it follows that, whatever positive number τ^2 stands for, $\frac{d\text{V}}{dp}$ is negative for all values of p that make $3+2p-p^2$ positive, that is, for all values of p less than 3. This Ivory gives, and so far he is correct; $\frac{d\text{V}}{dp}$ is certainly negative, and not zero, so long as p is less than 3. Ivory wishes to show that $\frac{d\text{V}}{dp}$ can never be zero. He takes the differential equation

$$\frac{d\text{V}}{dp}dp+\frac{d\text{V}}{\tau d\tau}\tau d\tau=0;$$

he says, "Further, in the differential equation $\frac{d\text{V}}{dp}$ cannot be zero; because, τ^2 increasing without limit, $\frac{d\text{V}}{\tau d\tau}\tau d\tau$ is essentially positive."

This is quite unsound. Regard p as the abscissa and τ as the corresponding ordinate of a curve determined by $\text{V}=0$; then assuming that $\frac{d\text{V}}{d\tau}$ is always positive, yet $\frac{d\text{V}}{dp}$ may vanish; that is, there may be a point or points on the curve at which the tangent is parallel to the axis of abscissæ. Suppose, for example, that V stood for

$$\tau^3-p^3+4ap^2-5a^2p-b^3;$$

then $\frac{d\text{V}}{d\tau}$ is always positive, $\frac{d\text{V}}{dp}$ is negative when p is less than a, and vanishes when $p=a$ or $\frac{5a}{3}$.

I do not assert that $\frac{d\text{V}}{dp}$ can vanish in the present case; I only maintain that Ivory's argument to show that $\frac{d\text{V}}{dp}$ cannot vanish is unsound. I shall presently demonstrate that $\frac{dV}{dp}$ cannot vanish.

20. Ivory concludes the two pages in the Philosophical Transactions for 1839 thus:—

"If the sign of $\frac{dV}{dp}$ be changed, the result will be positive; and hence, observing that x is contained between 0 and 1, we obtain a condition between any two values of p and r^2 that satisfy the equation (7), namely, the expression

$$(3+p-p^2)(1+p)+2pr^2$$

must be a positive quantity, or, which is the same thing,

$$r^2>(p^2-p+3)\frac{1+p}{2p}."$$

This is obscure in its commencement; but the statement to which it proceeds is intelligible, but is unwarranted. For, granting that $\frac{dV}{dp}$ is always to be negative, this will be secured if

$$(3+2p-p^2x^2)(1+px^2)+2pr^2x^2$$

is always positive; that is, if

$$3+2p+(3p+p^2+2pr^2)x^2-p^3x^4$$

is always positive; that is, if

$$\frac{3+2p}{x^2}+3p+p^2+2pr^2-p^3x^2$$

is always positive. And as x lies between 0 and 1, this condition is certainly secured if

$$3+2p+3p+p^2+2pr^2 \text{ is greater than } p^3.$$

According to Ivory's statement it would be *necessary* that

$$3+4p+2pr^2 \text{ should be greater than } p^3;$$

whereas we see that it would be *sufficient* that

$$3+5p+p^2+2pr^2 \text{ should be greater than } p^3.$$

Ivory's second statement is inconsistent with his first, but becomes consistent with it if we change $+3$ to -3; if, however, his second statement is to be taken as what he intended, and his first statement corrected to agree with it, his error is aggravated.

In fact, however, I doubt whether any such *necessary* criterion as Ivory proposes can be easily deduced from the value of $\frac{dV}{dp}$. For, granting that $\frac{dV}{dp}$ is always negative and never zero, it will not follow that *every element* in the integral which expresses $\frac{dV}{dp}$ must be negative, but only that the aggregate of the positive elements, if such there be, should fall short of the aggregate of the negative elements. However, be this as it may, there can be no doubt that the specific criterion which Ivory proposes is quite unsupported by demonstration.

When p and τ are very large, the relation between them is approximately that given in Art. 15.

21. In Arts. 14 and 16 it is shown that for every given value of p greater than p_0, there is one, and only one, value of τ which will make V vanish. We shall now show that corresponding to every value of τ there is one, and only one, value of p which will make V vanish.

Whatever be the given value of τ, it is obvious that V is positive when $p=0$, and negative when p is large enough: thus there must be some intermediate value of p which makes V vanish.

We have then to show that there is only *one* such value; to show this we shall demonstrate the proposition which Ivory asserted on insufficient grounds, namely, that $\frac{dV}{dp}$ can never vanish simultaneously with V.

$$\frac{dV}{dp}=-\int_0^1 \frac{x^4(1-x^2)}{\Delta^5}\left\{3+2p+(3p+p^2+2p\tau^2)x^2-p^3x^4\right\}dx;$$

and we know that $\frac{dV}{dp}$ cannot vanish so long as p is less than 3. See Art. 19.

We shall first show that the limit $p=3$ may be changed to a larger limit.

If the integral

$$\int_0^1 x^4(1-x^2)\{3+2p+(3p+p^2+2p\tau^2)x^2-p^3x^4\}dx$$

is positive, the integral will also be positive when Δ^5 is introduced as a divisor under the integral sign; for by introducing this denominator we diminish all the elements of the integral, but we diminish every negative element in a higher degree than any positive element.

Now the value of the last integral is

$$\frac{2(3+2p)}{5\times 7}+\frac{2(3p+p^2+2p\tau^2)}{7\times 9}-\frac{2p^3}{9\times 11};$$

we are certain that this is positive if

$$\frac{2p\tau^2+p^2+3p}{7\times 9}+\frac{3+2p}{5\times 7} \text{ is not less than } \frac{p^3}{9\times 11},$$

that is, if

$$\tau^2+\frac{3+p}{2}+\frac{9(3+2p)}{10p} \text{ is not less than } \frac{7p^2}{22}.$$

On trial it will be found that this condition is satisfied, even when τ is zero, provided p be not greater than $4\frac{1}{3}$. Thus we have only to consider the case in which both p is greater than $4\frac{1}{3}$, and $r^2+\frac{3+p}{2}+\frac{9(3+2p)}{10p}$ is less than $\frac{7p^2}{22}$; and we have to show that $\frac{dV}{dp}$ is negative in this case. Put $p=\frac{1}{q}$; thus V becomes a function of q and r; and we have to show that $\frac{dV}{dq}$ cannot vanish when q lies between 0 and $\frac{3}{13}$, and $r^2+\frac{3q+1}{2q}+\frac{9(3q+2)}{10}$ is less than $\frac{7}{22q^2}$, so that $r^2q^2+\frac{(3q+1)q}{2}+\frac{9(3q+2)q^2}{10}$ is less than $\frac{7}{22}$. We have

$$V=q\int_0^1\frac{x^2(1-x^2)\,(q^2-x^2)dx}{D^3},$$

where D^2 stands for $q^2+2qx^2+x^4+q^2r^2x^2$; therefore

$$\frac{dV}{dq}=\frac{V}{q}+q\int_0^1\frac{x^2(1-x^2)}{D^3}\left[2q-\frac{3(q^2-x^2)\,\{q+(1+qr^2)x^2\}}{D^2}\right]dx.$$

Separate the integral into two parts, one extending between the limits $x=0$ and $x=q$, and the other between the limits $x=q$ and $x=1$. It is obvious that the second part is positive, so that we have only the first part to examine; this is

$$\int_0^q\frac{x^2(1-x^2)}{D^5}\left\{-q^3+(3q+q^2-r^2q^3)x^2+(3+2q+3qr^2)x^4\right\}dx,$$

and we shall show that this is positive.

Let Q stand for

$$-q^3+(3q+q^2-r^2q^3)x^2+(3+2q+3qr^2)x^4;$$

then we shall first show that

$$\int_0^q(1-x^2)Q\,dx$$

is positive; and next that

$$\int_0^q\frac{x^2(1-x^2)Q\,dx}{D^5}$$

is positive.

Now $\int_0^q(1-x^2)Q\,dx$ will certainly be positive if

$$\int_0^q(1-x^2)\{-q^3+(3q+q^2-r^2q^3)x^2+3qr^2x^4\}dx$$

is positive. The last integral is

$$-q^3\left(q-\frac{q^3}{3}\right)+\left(3q+q^2-r^2q^3\right)\left(\frac{q^3}{3}-\frac{q^5}{5}\right)+3qr^2\left(\frac{q^5}{5}-\frac{q^7}{7}\right),$$

that is,

$$\frac{q^5}{3}\left(1-\frac{4q}{5}-\frac{3q^2}{5}\right)+\frac{4q^6r^2}{15}\left(1-\frac{6q^2}{7}\right);$$

and this is positive, for q does not exceed $\frac{3}{13}$.

Now the coefficient of x^2 in Q is positive, since r^2q^3 is less than $\frac{q}{3}$; and thus Q cannot change sign more than once; it changes sign once, for it is negative when $x=0$, and positive when $x=q$. The factor $\frac{x^2}{D^5}$ will be found to *increase continually* as x increases from 0 to q. Hence in passing from $\int_0^q (1-x^2)Q\,dx$ to $\int_0^q \frac{x^2(1-x^2)Q\,dx}{D^5}$, a larger fraction is taken of every positive element than of any negative element; so that since the former integral has been shown to be positive, the latter is necessarily positive.

To demonstrate that $\frac{x^2}{D^5}$ continually increases as x increases from 0 to q, take the differential coefficient with respect to x; the sign of this differential coefficient is the same as that of

$$2xD^2-5x^2(2qx+q^2r^2x+2x^3),$$

that is, of

$$2D^2-5x^2(2q+q^2r^2+2x^2),$$

that is, of

$$2q^2-(6q+3q^2r^2)x^2-8x^4;$$

this is positive when $x=0$; when $x=q$ it becomes $q^2(2-6q-3q^2r^2-8q^2)$, which we shall now show to be positive. We have by supposition

$$3r^2q^2+\frac{3q}{2}+\frac{99q^2}{10}+\frac{81q^3}{10} \text{ less than } \frac{21}{22},$$

and

$$2-6q-3q^2r^2-8q^2=2-\frac{9q}{2}-\left(\frac{3q}{2}+3q^2r^2+8q^2\right);$$

and as q does not exceed $\frac{3}{13}$, this is greater than $2-\frac{27}{26}-\frac{21}{22}$, and so is positive. Hence we see that when $V=0$, the value of $\frac{dV}{dq}$ is necessarily positive, and cannot be zero.

22. We have shown in the preceding Article, that $\frac{dV}{dp}$ cannot vanish simultaneously with V; the demonstration is rather complex, and perhaps for this reason Ivory attempted to establish the proposition by unsound general reasoning. Liouville does not give the proposition, although it is naturally required to make the discussion of the admissible values of p and r complete.

23. Thus we may state the results of Arts. 14, 16, and 21 in the fol-

lowing manner:—If the curve determined by the equation $V=0$ be traced in the first quadrant, every straight line parallel to the axis of p meets the curve once, and only once, and every straight line parallel to the axis of τ, and at a greater distance from it than p_0, meets the curve once, and only once. As $\frac{dV}{d\tau}$ vanishes with τ, the curve meets the axis of p at right angles; and from Art. 15 it follows that when τ and p are indefinitely great, the angle which the tangent to the curve makes with the axis of p is very nearly a right angle. Thus, for small values of τ the curve is concave to the axis of p, and for very large values of τ the curve is convex to the axis of p; so that the curve must have a point or points of inflexion.

24. In the very careful account of Ivory's mathematical researches, which is given in the fourth volume of the 'Abstracts of the Papers . . . of the Royal Society,' it is said, with respect to Jacobi's theorem, "In a paper in the Transactions for 1838, Mr. Ivory has with great elegance demonstrated this theorem, and has given, with greater detail than its authors had entered on, several statements regarding the limitations of the proportions of the axes." The language is cautious, but seems to imply some suspicion with regard to the accuracy of the statements. As we have now seen, many of Ivory's statements are inaccurate, and others, though accurate, are based on unsound reasoning.

"On the Theory of Continuous Beams." By JOHN MORTIMER HEPPEL, M. Inst. C.E. Communicated by Prof. W. J. MACQUORN RANKINE. Received December 9, 1869*.

In venturing to present to the Royal Society a paper on a subject which has engaged the attention, more especially in France, of some of the most eminent engineers and writers on Mechanical Philosophy, the author feels it to be incumbent on him to state the nature of the claim to their attention which he hopes it may be found to possess in point of originality or improvement on the method of treatment.

To do this clearly, however, it will be necessary to advert to the principal steps by which progress in the knowledge of this subject has been made, both in France and in this country.

The theory of continuous beams appears to have first attracted attention in France about 1825, when a method of determining all the conditions of equilibrium of a straight beam of uniform section throughout, resting on any number of level supports at any distances apart, each span being loaded uniformly, but the uniform loads varying in any manner from one span to another, was investigated and published by M. Navier. This method, although perfectly exact for the assumed conditions, was objectionable from the great labour and intricacy of the calculations it entailed.

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* Read January 27, 1870. See vol. xviii. p. 176.